

p -ADIC AND ADELIC MINISUPERSPACE QUANTUM COSMOLOGY

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We consider the formulation and some elaboration of p -adic and adelic quantum cosmology. The adelic generalization of the Hartle-Hawking proposal does not work in models with matter fields. p -adic and adelic minisuperspace quantum cosmology is well defined as an ordinary application of p -adic and adelic quantum mechanics. It is illustrated by a few of cosmological models in one, two and three minisuperspace dimensions. As a result of p -adic quantum effects and the adelic approach, these models exhibit some discreteness of the minisuperspace and cosmological constant. In particular, discreteness of the de Sitter space and its cosmological constant is emphasized.

1. Introduction

The main task of quantum cosmology¹ is to describe the evolution of the universe in a the very early stage. At this stage, the universe is in a quantum state, which is described by a wave function. Usually one takes it that this wave function is complex-valued and depends on some real parameters. Since quantum cosmology is related to the Planck scale phenomena it is logical to reconsider its foundations. We will here maintain the standard point of view that the wave function takes complex values, but we will treat its arguments in a more complete way. Namely, we will regard space-time coordinates and matter fields to be adelic, i.e. they have real as well as p -adic properties simultaneously. This approach is motivated

by the following reasons: (i) the field of rational numbers Q , which contains all observational and experimental numerical data, is a dense subfield not only in the field of real numbers R but also in the fields of p -adic numbers Q_p (p is any prime number), (ii) there is a plausible analysis² within and over Q_p as well as that one related to R , (iii) general mathematical methods and fundamental physical laws should be invariant³ under an interchange of the number fields R and Q_p , (iv) there is a quantum gravity uncertainty^{3,4} Δx while measuring distances around the Planck length ℓ_0 ,

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm, \quad (1)$$

which restricts the priority of archimedean geometry based on real numbers and gives rise to employment of non-archimedean geometry related to p -adic numbers,³ (v) it seems to be quite reasonable to extend compact archimedean geometries by the nonarchimedean ones in the path integral method, and (vi) adelic quantum mechanics⁵ applied to quantum cosmology provides realization of all the above statements.

The successful application of p -adic numbers and adeles in modern theoretical and mathematical physics started in 1987, in the context of string amplitudes^{6,7} (for a review, see Refs. 2, 8 and 9). For a systematic research in this field it was formulated p -adic quantum mechanics^{10,11} and adelic quantum mechanics.^{5,12} They are quantum mechanics with complex-valued wave functions of p -adic and adelic arguments, respectively. In the unified form, adelic quantum mechanics contains ordinary and all p -adic quantum mechanics.

As there is not an appropriate p -adic Schrödinger equation, there is also no p -adic generalization of the Wheeler-De Witt equation. Instead of the differential approach, Feynman's path integral method will be exploited.

p -adic gravity and the wave function of the universe were considered in the paper¹³ published in 1991. An idea of the fluctuating number fields at the Planck scale was introduced and it was suggested that we restrict the Hartle-Hawking¹⁴ proposal to the summation only over algebraic manifolds. It was shown that the wave function for the de Sitter minisuperspace model can be treated in the form of an infinite product of p -adic counterparts.

Another approach to quantum cosmology, which takes into account p -adic effects, was proposed in 1995.¹⁵ Like in adelic quantum mechanics, the adelic eigenfunction of the universe is a product of the corresponding eigenfunctions of real and all p -adic cases. p -adic wave functions are defined by p -adic generalization of the Hartle-Hawking¹⁴ path integral proposal. It was shown that in the framework of this procedure one obtains an adelic wave function for the de Sitter minisuperspace model. However, the adelic generalization with the Hartle-Hawking p -adic prescription does not work well when minisuperspace has more than one dimension, in particular, when matter fields are taken into consideration. The solution of this problem was found¹⁶ by treating minisuperspace cosmological models as models of adelic quantum mechanics.

In this paper we consider adelic quantum cosmology as an application of adelic quantum mechanics to the minisuperspace models. It will be illustrated by one-, two- and three-dimensional minisuperspace models. As a result of p -adic effects and the adelic approach, in these models there is some discreteness of minisuperspace and cosmological constant. This kind of discreteness was obtained for the first time in the context of adelic the de Sitter quantum model.¹⁵

In the next section we give some basic facts on p -adic and adelic mathematics. Section 3 is devoted to a brief review of p -adic and adelic quantum mechanics. p -adic and adelic quantum cosmology are formulated in Sec. 4. Sections 5 and 6 contain some concrete minisuperspace models. At the end, we give some concluding remarks.

2. p -Adic Numbers and Adeles

We give here a brief survey of some basic properties of p -adic numbers and adeles, which we exploit in this work.

Completion of \mathbb{Q} with respect to the standard absolute value $(|\cdot|_\infty)$ gives \mathbb{R} , and an algebraic extension of \mathbb{R} makes \mathbb{C} . According to the Ostrowski theorem¹⁷ any non-trivial norm on the field of rational numbers \mathbb{Q} is equivalent to the absolute value $|\cdot|_\infty$ or to a p -adic norm $|\cdot|_p$, where p is a prime number. p -adic norm is the non-archimedean (ultrametric) one and for a rational number $0 \neq x \in \mathbb{Q}$, where $x = p^\nu \frac{m}{n}$, $0 \neq n, \nu, m \in \mathbb{Z}$ and m, n are not divisible by p , has a value $|x|_p = p^{-\nu}$. Completion of \mathbb{Q} with respect to the p -adic norm for a fixed p leads to the corresponding field of p -adic numbers \mathbb{Q}_p . Completions of \mathbb{Q} with respect to $|\cdot|_\infty$ and all $|\cdot|_p$ exhaust all possible completions of \mathbb{Q} .

A p -adic number $x \in \mathbb{Q}_p$, in the canonical form, is an infinite expansion

$$x = p^\nu \sum_{i=0}^{+\infty} x_i p^i, \quad \nu, x_i \in \mathbb{Z}, \quad 0 \leq x_i \leq p-1. \quad (2)$$

The norm of p -adic number x in (2) is $|x|_p = p^{-\nu}$ and satisfies not only the triangle inequality, but also the stronger one

$$|x + y|_p \leq \max(|x|_p, |y|_p). \quad (3)$$

Metric on \mathbb{Q}_p is defined by $d_p(x, y) = |x - y|_p$. This metric is the non-archimedean one and the pair (\mathbb{Q}_p, d_p) presents locally compact, topologically complete, separable and totally disconnected p -adic metric space.

In the metric space \mathbb{Q}_p , p -adic ball $B_\nu(a)$, with the centre at the point a and the radius p^ν is the set

$$B_\nu(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\nu, \nu \in \mathbb{Z}\}. \quad (4)$$

The p -adic sphere $S_\nu(a)$ with the centre a and the radius p^ν is

$$S_\nu(a) = \{x \in \mathbb{Q}_p : |x - a|_p = p^\nu, \nu \in \mathbb{Z}\}. \quad (5)$$

The following holds:

$$\begin{aligned}
B_\nu(a) &= \bigcup_{\nu' \leq \nu} S_{\nu'}(a), \\
S_\nu(a) &= B_\nu(a) \setminus B_{\nu-1}(a), \quad B_\nu(a) \subset B_{\nu'}(a), \quad \nu < \nu', \\
\bigcap_\nu B_\nu(a) &= \{a\}, \quad \bigcup_\nu B_\nu(a) = \bigcup_\nu S_\nu(a) = Q_p.
\end{aligned} \tag{6}$$

Elementary p -adic functions¹⁸ are given by the series of the same form as in the real case, e.g.

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{7}$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!}, \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \tag{8}$$

$$\tanh x = \sum_{k=2}^{\infty} \frac{2^k(2^k-1)B_k x^{k-1}}{k!}, \quad \coth x = \frac{1}{x} + \sum_{k=2}^{\infty} \frac{2^k B_k x^{k-1}}{k!}, \tag{9}$$

where B_k are Bernoulli's numbers. These functions have the same domain of convergence $G_p = \{x \in Q_p : |x|_p < |2|_p\}$. Note the following p -adic norms of the hyperbolic functions: $|\sinh x|_p = |x|_p$ and $|\cosh x|_p = 1$.

Real and p -adic numbers are unified in the form of the adeles.¹⁹ An adele is an infinite sequence

$$a = (a_\infty, a_2, \dots, a_p, \dots), \tag{10}$$

where $a_\infty \in Q_\infty$, and $a_p \in Q_p$, with restriction that $a_p \in Z_p$ ($Z_p = \{x \in Q_p : |x|_p \leq 1\}$) for almost all p , i.e. for all but a finite set S of primes p .

If we introduce $\mathcal{A}(S) = Q_\infty \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p$, then the space of all adeles is $\mathcal{A} = \bigcup_S \mathcal{A}(S)$, which is a topological ring. Namely, \mathcal{A} is a ring with respect to the componentwise addition and multiplication. A principal adele is a sequence $(r, r, \dots, r, \dots) \in \mathcal{A}$, where $r \in Q$. Thus, the ring of principal adeles, which is a subring of \mathcal{A} , is isomorphic to Q .

An important function on \mathcal{A} is the additive character $\chi(x)$, $x \in \mathcal{A}$, which is a continuous and complex-valued function with basic properties:

$$|\chi(x)|_\infty = 1, \quad \chi(x+y) = \chi(x)\chi(y). \tag{11}$$

This additive character may be presented as

$$\chi(x) = \prod_v \chi_v(x_v) = \exp(-2\pi i x_\infty) \prod_p \exp(2\pi i \{x_p\}_p), \tag{12}$$

where $v = \infty, 2, \dots, p, \dots$, and $\{x\}_p$ is the fractional part of the p -adic number x .

Map $\varphi : \mathcal{A} \rightarrow C$, which has the form

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_{p \in S} \varphi_p(x_p) \prod_{p \notin S} \Omega(|x_p|_p), \tag{13}$$

where $\varphi_\infty(x_\infty)$ is an infinitely differentiable function on Q_∞ and falls to zero faster than any power of $|x_\infty|_\infty$ as $|x_\infty|_\infty \rightarrow \infty$, $\varphi_p(x_p)$ is a locally constant function with compact support, and

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1, \end{cases} \quad (14)$$

is called an elementary function of \mathcal{A} . Finite linear combinations of elementary functions (13) make the set of the Schwartz-Bruhat functions¹⁹ $D(\mathcal{A})$. The existence of Ω -function is unavoidable for a construction of any adelic model. The Fourier transform is

$$\tilde{\varphi}(\xi) = \int_{\mathcal{A}} \varphi(x) \chi(\xi x) dx \quad (15)$$

and it maps one-to-one $D(\mathcal{A})$ onto $D(\mathcal{A})$. It is worth noting that Ω -function is a counterpart of the Gaussian $\exp(-\pi x^2)$ in the real case, since it is invariant with respect to the Fourier transform.

The integrals² of the Gauss type over the p -adic sphere $S_\nu = S_\nu(0)$, p -adic ball $B_\nu = B_\nu(0)$ and over any Q_v are:

$$\int_{S_\nu} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} \lambda_p(\alpha) |2\alpha|_p^{-1/2} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), & \left|\frac{\beta}{2\alpha}\right|_p = p^\nu, \\ 0, & \left|\frac{\beta}{2\alpha}\right|_p \neq p^\nu, \end{cases} \quad (16)$$

for $|4\alpha|_p \geq p^{2-2\nu}$,

$$\int_{B_\nu} \chi_p(\alpha x^2 + \beta x) dx = \begin{cases} p^\nu \Omega(p^\nu |\beta|_p), & |\alpha|_p p^{2\nu} \leq 1, \\ \frac{\lambda_p(\alpha)}{|2\alpha|_p^{1/2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right) \Omega\left(p^{-\nu} \left|\frac{\beta}{2\alpha}\right|_p\right), & |4\alpha|_p p^{2\nu} > 1, \end{cases} \quad (17)$$

$$\int_{Q_v} \chi_p(\alpha x^2 + \beta x) dx = \lambda_v(\alpha) |2\alpha|_v^{-1/2} \chi_v\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0. \quad (18)$$

The arithmetic functions $\lambda_v(x) : Q_v \mapsto C$, where $v = \infty, 2, 3, 5, \dots$, are defined² as follows: $\lambda_v(0) = 1$, $\lambda_\infty(x) = \frac{1}{\sqrt{2}}(1 - i \operatorname{sign} x)$,

$$\lambda_p(x) = \begin{cases} 1, & \nu = 2k, \quad p \neq 2, \\ \left(\frac{x_0}{p}\right), & \nu = 2k+1, \quad p \equiv 1 \pmod{4}, \\ i\left(\frac{x_0}{p}\right), & \nu = 2k+1, \quad p \equiv 3 \pmod{4}, \end{cases} \quad (19)$$

$$\lambda_2(x) = \begin{cases} \frac{1}{\sqrt{2}}[1 + (-1)^{x_1} i], & \nu = 2k, \\ \frac{1}{\sqrt{2}}(-1)^{x_1+x_2}[1 + (-1)^{x_1} i], & \nu = 2k+1, \end{cases}$$

where p -adic x is given by (2), $k \in \mathbb{Z}$, and $\left(\frac{x_0}{p}\right)$ is the Legendre symbol. However we will mainly use their properties:

$$|\lambda_v(a)|_\infty = 1, \quad \lambda_v(ab^2) = \lambda_v(a), \quad \lambda_v(a)\lambda_v(b) = \lambda_v(a+b)\lambda_v(ab(a+b)). \quad (20)$$

3. p -Adic and Adelic Quantum Mechanics

In foundations of standard quantum mechanics (over R) one usually starts with a representation of the canonical commutation relation

$$[\hat{q}, \hat{k}] = i\hbar, \quad (21)$$

where q is a spatial coordinate and k is the corresponding momentum. It is well known that the procedure of quantization is not unique. In formulation of p -adic quantum mechanics^{10,11} the multiplication $\hat{q}\psi \rightarrow x\psi$ has no meaning for $x \in Q_p$ and $\psi(x) \in C$. Also, there is no possibility to define p -adic "momentum" or "Hamiltonian" operator. In the real case they are infinitesimal generators of space and time translations, but, since Q_p is disconnected field, these infinitesimal transformations become meaningless. However, finite transformations remain meaningful and the corresponding Weyl and evolution operators are p -adically well defined.

The canonical commutation relation in the p -adic case can be represented by the Weyl operators ($\hbar = 1$)

$$\hat{Q}_p(\alpha)\psi_p(x) = \chi_p(\alpha x)\psi_p(x), \quad (22)$$

$$\hat{K}_p(\beta)\psi(x) = \psi_p(x + \beta). \quad (23)$$

Now, instead of the relation (21) in the real case, we have

$$\hat{Q}_p(\alpha)\hat{K}_p(\beta) = \chi_p(\alpha\beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha) \quad (24)$$

in the p -adic one. It is possible to introduce the product of unitary operators

$$\hat{W}_p(z) = \chi_p(-\frac{1}{2}qk)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad z \in Q_p \times Q_p, \quad (25)$$

that is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements (z, α) with the group product

$$(z, \alpha) \cdot (z', \alpha') = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z')), \quad (26)$$

where $B(z, z') = -kq' + qk'$ is a skew-symmetric bilinear form on the phase space. Dynamics of a p -adic quantum model is described by a unitary evolution operator $U(t)$ without using the Hamiltonian operator. Instead of that, the evolution operator has been formulated in terms of its kernel $\mathcal{K}_t(x, y)$

$$U_p(t)\psi(x) = \int_{Q_p} \mathcal{K}_t(x, y)\psi(y)dy. \quad (27)$$

In this way,¹⁰ p -adic quantum mechanics is given by a triple

$$(L_2(Q_p), W_p(z_p), U_p(t_p)). \quad (28)$$

Keeping in mind that standard quantum mechanics can be also given as the corresponding triple, ordinary and p -adic quantum mechanics can be unified in the form of adelic quantum mechanics^{5,12}

$$(L_2(\mathcal{A}), W(z), U(t)). \quad (29)$$

$L_2(\mathcal{A})$ is the Hilbert space on \mathcal{A} , $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathcal{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathcal{A})$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_{\mathcal{A}} \mathcal{K}_t(x, y)\psi(y)dy = \prod_v \int_{Q_v} \mathcal{K}_t^{(v)}(x_v, y_v)\psi^{(v)}(y_v)dy_v. \quad (30)$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x), \quad (31)$$

where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = (E_\infty, E_2, \dots, E_p, \dots)$ is the corresponding adelic energy, indices α and β denote energy levels and their degeneration. Any adelic eigenfunction has the form

$$\Psi_S(x) = \Psi_\infty(x_\infty) \prod_{p \in S} \Psi_p(x_p) \prod_{p \notin S} \Omega(|x_p|_p), \quad x \in \mathcal{A}, \quad (32)$$

where $\Psi_\infty \in L_2(R)$, $\Psi_p \in L_2(Q_p)$ are ordinary and p -adic eigenfunctions, respectively. The Ω -function is defined by (14), it is an element of the Hilbert space $L_2(Q_p)$, and provides convergence of the infinite product (32). Note that (32) has the same form as (13), but here all factors are elements of the Hilbert spaces on R and Q_p of the same quantum system. In an adelic eigenstate, p -adic eigenstates are $\Omega(|x_p|_p)$ for all but a finite set S of primes p . Hence, the existence of $\Omega(|x_p|_p)$ for all or almost all p is a necessary condition for a quantum model to be adelic one.

For a fixed S , function $\Psi_S(x)$ in (32) may be regarded as an element of the Hilbert space $L_2(\mathcal{A}(S))$, where $\mathcal{A}(S)$ is a subset of adeles \mathcal{A} defined in Sec. 2. Moreover, $\Psi_\infty(x_\infty)$ and $\Psi_p(x_p)$ ($p \in S$) may be not only eigenstates but also any element of $L_2(R)$ and $L_2(Q_p)$, respectively. Then superposition $\Psi(x) = \sum_S C(S)\Psi_S(x)$, where $\sum_S |C(S)|_\infty^2 = 1$ and $\Psi_S \in L_2(\mathcal{A}(S))$, is an element of $L_2(\mathcal{A})$.

A suitable way to calculate p -adic propagator $\mathcal{K}_p(x'', t''; x', t')$ is to use Feynman's path integral method, i.e.

$$\mathcal{K}_p(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_p \left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) \mathcal{D}q. \quad (33)$$

For quadratic Lagrangians it has been evaluated^{21,22} in the same way for real and p -adic cases, and the following exact general expression is obtained:

$$\mathcal{K}_v(x'', t''; x', t') = \lambda_v \left(-\frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right) \left| \frac{1}{h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right|_v^{\frac{1}{2}} \chi_v \left(-\frac{1}{h} \bar{S}(x'', t''; x', t') \right), \quad (34)$$

where λ_v functions satisfy properties (19) and (20), and $\bar{S}(x'', t''; x', t')$ is the classical action. When one has a system with more than one dimension with uncoupled spatial coordinates, the total propagator is the product of the corresponding one-dimensional propagators. As an illustration of p -adic and adelic quantum-mechanical models, the following one-dimensional systems with the quadratic Lagrangians were considered: A free particle and a harmonic oscillator,^{2,5,12} a particle in a constant field,²³ a free relativistic particle²⁰ and a harmonic oscillator with time-dependent frequency.²⁴

Adelic quantum mechanics takes into account ordinary as well as p -adic quantum effects and may be regarded as a starting point for the construction of a more complete superstring and M-theory. In the low-energy limit adelic quantum mechanics becomes the ordinary one.²⁰

4. p -Adic and Adelic Quantum Cosmology

Any real space-time manifold in standard quantum cosmology contains rational points which are dense in the field of real numbers. These rational points, or some of them, may be completed with respect to a distance induced by p -adic norm on Q and one obtains p -adic counterpart of this real manifold. Since this can be done for every p , in this way, we get an infinite ensemble of (real and p -adic) manifolds, which in the form of the direct product usually make an adelic space-time. This adelic space-time provides an arena for a simultaneous exhibition of real and p -adic aspects of gravitational and matter fields of the same quantum cosmological model. According to the motivations (i)-(vi) stated in Introduction, it is quite reasonable to consider our very early universe as an adelic quantum system.

Adelic quantum cosmology is an application of adelic quantum theory to the universe as a whole.¹⁵ Adelic quantum theory unifies both p -adic and standard quantum theory.⁵ In the path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method, i.e. the amplitude to go from one state with intrinsic metric h'_{ij} and matter configuration ϕ' on an initial hypersurface Σ' to another state with metric h''_{ij} and matter configuration ϕ'' on a final hypersurface Σ'' is given by a functional integral

$$\langle h''_{ij}, \phi'', \Sigma'' | h'_{ij}, \phi', \Sigma' \rangle_\infty = \int \mathcal{D}(g_{\mu\nu})_\infty \mathcal{D}(\Phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]), \quad (35)$$

over all four-geometries $g_{\mu\nu}$ and matter configurations Φ , which interpolate between the initial and final configurations.¹ In this expression $S[g_{\mu\nu}, \Phi]$ is an Einstein-Hilbert action for the gravitational and matter fields. This action can be calculated if we use metric in the standard 3+1 decomposition

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j, \quad (36)$$

where N and N_i are the lapse and shift functions.

To perform p -adic and adelic generalization we first make p -adic counterpart of the action using form-invariance under change of real to the p -adic number fields.

Then we generalize (35) and introduce p -adic complex-valued cosmological amplitude

$$\langle h''_{ij}, \phi'', \Sigma'' | h'_{ij}, \phi', \Sigma' \rangle_p = \int \mathcal{D}(g_{\mu\nu})_p \mathcal{D}(\Phi)_p \chi_p(-S_p[g_{\mu\nu}, \Phi]), \quad (37)$$

where $g_{\mu\nu}(x)$ and $\Phi(x)$ are the corresponding p -adic counterparts of metric and matter fields continually connecting their values on Σ' and Σ'' . In its general aspects p -adic functional integral (37) mimics the usual Feynman path integral (for one-dimensional case, see Refs. 2 and 21). The definite integral in the classical action is understood as the usual difference of the indefinite one (without pseudoconstants) at final and initial points. The measures $\mathcal{D}(g_{\mu\nu})_p$ and $\mathcal{D}(\Phi)_p$ are related to the real-valued Haar measure on p -adic spaces, and the path integral is the limit of a k -multiple integral when $k \rightarrow \infty$. There is no natural ordering on Q_p , but one can define an appropriate linear order.²

Note that in (35) and (37) one has to take also a sum over manifolds which have Σ'' and Σ' as their boundaries. Since the problem of topological classification of four-manifolds is algorithmically unsolvable it was proposed¹³ that summation should be taken over algebraic manifolds. Our adelic approach supports this proposal, since algebraic manifolds maintain all rational points under the interchange of number fields R and Q_p .

Since the space of all three-metrics and matter field configurations on a three-surface, called superspace, has infinitely many dimensions, one takes an approximation. A useful approximation is to truncate the infinite degrees of freedom to a finite number $q_\alpha(t)$, ($\alpha = 1, 2, \dots, n$). In this way, one obtains a particular minisuperspace model. Usually, one restricts the four-metric to be of the form (36), with $N^i = 0$ and h_{ij} as functions $q_\alpha(t)$. For the homogeneous and isotropic cosmologies, the usual metric is a Robertson-Walker one, of which the spatial sector has the form

$$h_{ij} dx^i dx^j = a^2(t) d\Omega_3^2 = a^2(t) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)]. \quad (38)$$

If we use also a single scalar field ϕ , as a matter content of the model, minisuperspace coordinates are $\{a, \phi\}$. More generally, models can be homogeneous but also anisotropic ones, and they will be here also considered. All such models can be classified as: (i) Kantowski-Sachs models with spatial topology $S^1 \times S^2$ and

$$h_{ij} dx^i dx^j = a^2(t) dr^2 + b^2(t) d\Omega_2^2, \quad (39)$$

where $d\Omega_2^2$ is the metric on the two-sphere, and minisuperspace coordinates are $\{a, b, \phi\}$; (ii) Bianchi models, which are the most general homogeneous cosmological models with a three-dimensional group of isometries. The three-metric of each of these models can be written in the form $h_{ij} dx^i dx^j = h_{ij}(t) \omega^i \otimes \omega^j$, where ω^i are the invariant one-forms associated with the isometry group. The simplest example is the Bianchi I model with $\omega^1 = dx$, $\omega^2 = dy$ and $\omega^3 = dz$, and

$$h_{ij} dx^i dx^j = a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2, \quad (40)$$

where minisuperspace coordinates are $\{a, b, c, \phi\}$. For the minisuperspace models, functional integrals in (35) and (37) are reduced to functional integrals over three-metric and configuration of matter fields, and to another usual integral over the lapse function N . For the boundary condition $q_\alpha(t'') = q'_\alpha$, $q_\alpha(t') = q'_\alpha$ in the gauge $\dot{N} = 0$, we have the v -adic minisuperspace propagator

$$\langle q''_\alpha | q'_\alpha \rangle_v = \int_{G_v} dN \mathcal{K}_v(q''_\alpha, N; q'_\alpha, 0), \quad (41)$$

where G_v is specified according to the adelic approach, i.e. $G_\infty = R$ and $G_p = Z_p$ for all or almost all p , and

$$\mathcal{K}_v(q''_\alpha, N; q'_\alpha, 0) = \int \mathcal{D}q_\alpha \chi_v(-S_v[q_\alpha]), \quad (42)$$

is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates (q'_α, q''_α) in a fixed 'time' N . S_v is the v -adic action of the minisuperspace model, i.e.

$$S_v[q_\alpha] = \int_0^1 dt N \left[\frac{1}{2N^2} f_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta - U(q) \right], \quad (43)$$

where $f_{\alpha\beta}$ is a minisuperspace metric ($ds_m^2 = f_{\alpha\beta} dq^\alpha dq^\beta$) with an indefinite signature $(-, +, +, \dots)$. This metric includes spatial (gravitational) components and also matter variables for the given model. It is worth emphasizing that in the adelic approach the lapse function N and minisuperspace coordinates q_α have adelic structure. Also, constants and parameters must be the same rational numbers in R and all Q_p .

The standard minisuperspace ground state wave function in the Hartle-Hawking (no-boundary) proposal,¹⁴ will be attained if one performs a functional integration in the Euclidean version of

$$\Psi_\infty[h_{ij}] = \int \mathcal{D}(g_{\mu\nu})_\infty \mathcal{D}(\Phi)_\infty \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]), \quad (44)$$

over all compact four-geometries $g_{\mu\nu}$ which induce h_{ij} at the compact three-manifold. This three-manifold is the only boundary of all the four-manifolds. If we generalize the Hartle-Hawking proposal to the p -adic minisuperspace, then an adelic Hartle-Hawking wave function is an infinite product

$$\Psi[h_{ij}] = \prod_v \int \mathcal{D}(g_{\mu\nu})_v \mathcal{D}(\Phi)_v \chi_v(-S_v[g_{\mu\nu}, \Phi]), \quad (45)$$

where the path integration must be performed over both, archimedean and nonarchimedean geometries. If after evaluation of the corresponding functional integrals we obtain as a result $\Psi[h_{ij}]$ in the form (32), we will say that such cosmological model is an adelic one.

As we shall see, a more successful p -adic generalization of the minisuperspace cosmological models can be performed in the framework of p -adic and adelic quantum mechanics¹⁶ without using the Hartle-Hawking proposal. In such cases, we

examine the conditions under which some eigenstates of the evolution operator (31) exist.

5. p -Adic Models in the Hartle-Hawking Proposal

The Hartle-Hawking proposal for the wave function of the universe is generalized to p -adic case in Refs. 15 and 25. In this approach, p -adic wave function is given by the integral

$$\Psi_p(q^\alpha) = \int_{G_p} dN \mathcal{K}_p(q^\alpha, N; 0, 0), \quad (46)$$

where, according to the adelic structure of N , $G_p = Z_p$ (i.e. $|N|_p \leq 1$) for every or almost every p .

5.1. Models of the de Sitter type

Models of the de Sitter type are models with cosmological constant Λ and without matter fields. We consider two minisuperspace models of this type, with $D = 4$ and $D = 3$ space-time dimensions. The corresponding real Einstein-Hilbert action is²⁶

$$S = \frac{1}{16\pi G} \int_M d^D x \sqrt{-g} (R - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial M} d^{D-1} x \sqrt{h} K, \quad (47)$$

where R is the scalar curvature of D -dimensional manifold M , Λ is the cosmological constant, and K is the trace of the extrinsic curvature K_{ij} on the boundary ∂M . The metric for this model²⁶ is of the Robertson-Walker type

$$ds^2 = \sigma^{D-2} [-N^2 dt^2 + a^2(t) d\Omega_{D-1}^2]. \quad (48)$$

In this expression $d\Omega_{D-1}^2$ denotes the metric on the unit $(D-1)$ -sphere, $\sigma^{D-2} = 8\pi G / [V^{D-1}(D-1)(D-2)]$, where V^{D-1} is the volume of the unit $(D-1)$ -sphere.

5.1.1. The de Sitter model in $D = 3$ dimensions

In the real $D = 3$ case, the model is related to the multiple-sphere configuration and wormhole solutions.²⁶ v -adic classical action for this model is

$$\bar{S}_v(a'', N; a', 0) = \frac{1}{2\sqrt{\lambda}} \left[N\sqrt{\lambda} + \lambda \left(\frac{2a''a'}{\sinh(N\sqrt{\lambda})} - \frac{a'^2 + a''^2}{\tanh(N\sqrt{\lambda})} \right) \right]. \quad (49)$$

Let us note that λ , ($\lambda = \Lambda G^2$), denotes the rescaled cosmological constant Λ . Using (34) for the propagator of this model we have

$$\mathcal{K}_v(a'', N; a', 0) = \lambda_v \left(-\frac{2\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right) \left| \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right|_v^{1/2} \chi_v(-\bar{S}_v(a'', N; a', 0)). \quad (50)$$

The p -adic Hartle-Hawking wave function is

$$\Psi_p(a) = \int_{|N|_p \leq 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda} \coth(N\sqrt{\lambda})}{2} a^2 \right), \quad (51)$$

which after p -adic integration becomes

$$\Psi_p(a) = \begin{cases} \Omega(|a|_p), & |\lambda|_p \leq p^{-2}, \quad p \neq 2, \\ \frac{1}{2}\Omega(|a|_2), & |\lambda|_2 \leq 2^{-4}, \quad p = 2. \end{cases} \quad (52)$$

5.1.2. The de Sitter model in $D = 4$ dimensions

The de Sitter model in $D = 4$ space-time dimensions may be described by the metric²⁷

$$ds^2 = \sigma^2 \left(-\frac{N^2}{q(t)} dt^2 + q(t) d\Omega_3^2 \right), \quad \sigma^2 = \frac{2G}{3\pi}, \quad (53)$$

and the corresponding action $S_v[q] = \frac{1}{2} \int_{t'}^{t''} dt N \left(-\frac{\dot{q}^2}{4N^2} - \lambda q + 1 \right)$, where $\lambda = 2\Lambda G/(9\pi)$. For $N = 1$, the equation of motion $\ddot{q} = 2\lambda$ has solution $q(t) = \lambda t^2 + \left(\frac{q'' - q'}{T} - \lambda T \right) t + q'$, where $q'' = q(t'')$, $q' = q(t')$ and $T = t'' - t'$. Note that this classical solution resembles motion of a particle in a constant field and defines an algebraic manifold. The choice of metric in the form (53) yields quadratic v -adic classical action

$$\bar{S}_v(q'', T; q', 0) = \frac{\lambda^2 T^3}{24} - [\lambda(q' + q'') - 2] \frac{T}{4} - \frac{(q'' - q')^2}{8T}. \quad (54)$$

According to (34), the corresponding propagator is

$$\mathcal{K}_v(q'', T|q', 0) = \frac{\lambda_v(-8T)}{|4T|_v^{1/2}} \chi_v(-\bar{S}_v(q'', T|q', 0)). \quad (55)$$

We obtain the p -adic Hartle-Hawking wave function by the integral

$$\Psi_p(q) = \int_{|T|_p \leq 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p \left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2) \frac{T}{4} + \frac{q^2}{8T} \right), \quad (56)$$

and as a result we get^{15,25} also $\Omega(|q|_p)$ function with the condition $\lambda = 4 \cdot 3 \cdot l$, $l \in Z_p$.

The above Ω -functions allow adelic wave functions of the form (32) for both $D = 3$ and $D = 4$ cases. Since $|\lambda|_p \leq p^{-2}$ in (52) for all $p \neq 2$, it means that λ cannot be a rational number and consequently the above the de Sitter minisuperspace model in $D = 3$ space-time dimensions is not adelic one. However $D = 4$ case is adelic, because $\lambda = 4 \cdot 3 \cdot l$ is a rational number when $l \in Z \subset Z_p$.

5.2. Model with a homogeneous scalar field

To deal with the models of the de Sitter type is very instructive. Although these models are without matter content, they are in quantum cosmology of such significance as the model of harmonic oscillator in quantum mechanics. However, it is also important to consider models with some matter content. In order to have a quadratic classical action, we use metric in the form²⁸

$$ds^2 = \sigma^2 \left(-N^2(t) \frac{dt^2}{a^2(t)} + a^2(t) d\Omega_3^2 \right), \quad (57)$$

the gravitational part of the action in the form (47) (with $D = 4$), and the corresponding action for a scalar field as

$$S_{matter} = -\frac{1}{2} \int_M d^4x \sqrt{-g} [g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi)]. \quad (58)$$

After substitutions: $\Phi = \sqrt{3/(4\pi G)}\phi$, $V(\phi) = \alpha \cosh(2\phi) + \beta \sinh(2\phi)$ and $x = a^2 \cosh(2\phi)$, $y = a^2 \sinh(2\phi)$, we get the classical action and propagator

$$\begin{aligned} \bar{S}_p(x'', y'', N | x', y', 0) = & \frac{\alpha^2 - \beta^2}{24} N^3 + \frac{1}{4} (2 - \alpha(x' + x'') - \beta(y' + y'')) N \\ & + \frac{(y'' - y')^2 - (x'' - x')^2}{8N}, \end{aligned} \quad (59)$$

$$\mathcal{K}_p(x'', y'', N | x', y', 0) = \frac{1}{|4N|_p} \chi_p(-\bar{S}_p(x'', y'', N; x', y', 0)). \quad (60)$$

As we have shown¹⁶ for this model, a p -adic Hartle-Hawking wave function in the form of Ω -function does not exist. This leads to the conclusion that either the above model is not adelic, or that p -adic generalization of the Hartle-Hawking proposal is not an adequate one. However, if in the action (59) we take $\beta = 0$, $y = 0$, then we get classical action for the de Sitter model (54), and such model, as we showed it, is the adelic one. The similar conclusion holds also for some other models in which minisuperspace is not one-dimensional. This is a reason to regard p -adic and adelic minisuperspace quantum cosmology just as the corresponding application of p -adic and adelic quantum mechanics without the Hartle-Hawking proposal.

6. Minisuperspace Models in p -Adic and Adelic Quantum Mechanics

In this approach we investigate conditions under which quantum-mechanical p -adic ground state exists in the form of Ω -function and some other typical eigenfunctions. This leads to the desired result and it enables adelicization of many exactly soluble minisuperspace cosmological models, usually with some restrictions on the parameters of the models.

The necessary condition for the existence of an adelic quantum model is the existence of p -adic ground state $\Omega(|q_\alpha|_p)$ defined by (14), i.e.

$$\int_{|q_{\alpha'}|_p \leq 1} \mathcal{K}_p(q_\alpha'', N; q_{\alpha'}, 0) dq_{\alpha'} = \Omega(|q_\alpha''|_p). \quad (61)$$

Analogously, if a system is in the state $\Omega(p^\nu |q_\alpha|_p)$, where $\Omega(p^\nu |q_\alpha|_p) = 1$ if $|q_\alpha|_p \leq p^{-\nu}$ and $\Omega(p^\nu |q_\alpha|_p) = 0$ if $|q_\alpha|_p > p^{-\nu}$, then its kernel must satisfy equation

$$\int_{|q_{\alpha'}|_p \leq p^{-\nu}} \mathcal{K}_p(q_\alpha'', N; q_{\alpha'}, 0) dq_{\alpha'} = \Omega(p^\nu |q_\alpha''|_p). \quad (62)$$

If p -adic ground state is of the form of the δ -function, where δ -function is defined as $\delta(a - b) = 1$ if $a = b$ and $\delta(a - b) = 0$ if $a \neq b$, then the corresponding kernel of

the model has to satisfy equation

$$\int_{|q'_\alpha|_p=p^\nu} \mathcal{K}_p(q''_\alpha, T; q', 0) dq'_\alpha = \delta(p^\nu - |q''_\alpha|_p). \quad (63)$$

Equations (61) and (62) are usual p -adic vacuum ingredients of the adelic eigenvalue problem (31), i.e. $\int_{Q_p} \mathcal{K}_p(x'', t; x', 0) \Psi_p(x') dx' = \chi_p(Et) \Psi_p(x'')$, where $\chi_p(Et) = 1$ in the vacuum state $\{Et\}_p = 0$. The above Ω and δ functions do not make a complete set of p -adic eigenfunctions, but they are very simple and illustrative. Since these functions have finite supports, the ranges of integration in (61)-(63) are also finite. The lapse function N is under the kernel $\mathcal{K}(q''_\alpha, N; q'_\alpha, 0)$ and is restricted to some values on which eigenfunctions do not depend explicitly.

In the following, we apply (61)-(63) to some minisuperspace models.

6.1. Models of the de Sitter type

6.1.1. The de Sitter model in $D = 3$ dimensions

By application of the above exposed formalism of p -adic quantum mechanics in the form (6.1) and (6.2), for this model we found¹⁶ the ground state

$$\Psi_p(a) = \begin{cases} \Omega(|a|_p), & |N|_p \leq 1, \quad |\lambda|_p \leq \frac{1}{p^2}, \quad p \neq 2, \\ \Omega(|a|_2), & |N|_2 \leq \frac{1}{4}, \quad |\lambda|_2 \leq 4, \quad p = 2, \end{cases} \quad (64)$$

and also

$$\Psi_p(a) = \begin{cases} \Omega(p^\nu |a|_p), & |N|_p \leq p^{-2\nu}, \quad |\lambda|_p \leq p^{4\nu-2}, \quad p \neq 2, \\ \Omega(2^\nu |a|_2), & |N|_2 \leq 2^{-2-2\nu}, \quad |\lambda|_2 \leq 2^{4\nu+2}, \quad p = 2, \end{cases} \quad (65)$$

where $\nu = 1, 2, 3, \dots$. For simplicity, in the sequel the upper and lower row will be related to the $p \neq 2$ and $p = 2$ cases, respectively.

The existence of the ground state in the form of the δ -function may be investigated by the Eq. (63), i.e.

$$\int_{Q_p} \mathcal{K}_p(a'', N; a', 0) \delta(p^\nu - |a'|) da' = \delta(p^\nu - |a''|), \quad (66)$$

with the kernel (50), that leads to the equation

$$\begin{aligned} & \lambda_p \left(-\frac{\sqrt{\lambda}}{2 \sinh(N\sqrt{\lambda})} \right) \left| \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} \right|_p^{1/2} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}}{2 \tanh(N\sqrt{\lambda})} a''^2 \right) \\ & \times \int_{|a'|_p=p^\nu} \chi_p \left(\frac{\sqrt{\lambda}}{2 \tanh(N\sqrt{\lambda})} a'^2 - \frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})} a'' a' \right) da' = \delta(p^\nu - |a''|_p). \end{aligned} \quad (67)$$

The above integration is performed over p -adic sphere with the radius p^ν and for $|\frac{N}{2}|_p \leq p^{2\nu-2}$, $\nu = 1, 0, -1, -2, \dots$. As a result, on the left hand side we have

$$\chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}}{2} \tanh(N\sqrt{\lambda}) a''^2 \right).$$

To have an equality, the argument of the additive character must be equal or less than unity. This requirement leads to the condition

$$\left| \frac{\sqrt{\lambda} \tanh(N\sqrt{\lambda}) a''^2}{2} \right|_p \leq p^{4\nu-2} |\lambda|_p \leq 1, \quad |\lambda|_p \leq p^{2-4\nu}.$$

This (for the p -adic norms of N and λ) is also related to the domain of convergence of the analytic function $\tanh x$,

$$|N\sqrt{\lambda}|_p = |N|_p |\lambda|_p^{1/2} \leq p^{2\nu-2} \cdot p^{1-2\nu} = p^{-1}.$$

If $p = 2$, then condition $|N|_2 \leq 2^{2\nu-3}$ holds, for $\nu = 1, 0, -1, -2, \dots$, and we are in the domain of convergence. Finally, we also conclude that p -adic ground state

$$\Psi_p(a) = \begin{cases} \delta(p^\nu - |a|_p), & |N|_p \leq p^{2\nu-2}, \quad |\lambda|_p \leq p^{2-4\nu}, \\ \delta(2^\nu - |a|_2), & |N|_2 \leq 2^{2\nu-3}, \quad |\lambda|_2 \leq 2^{2-4\nu}, \end{cases} \quad (68)$$

exists for $\nu = 1, 0, -1, -2, \dots$.

This de Sitter model is p -adic but it is not an adelic one for the same reasons as in the above Hartle-Hawking approach.

6.1.2. The de Sitter model in $D = 4$ dimensions

Here we start using the Eqs. (53)-(55). As it was already shown,¹⁶ the ground states for this model exist in the forms

$$\Psi_p(q) = \begin{cases} \Omega(|q|_p), & |T|_p \leq 1, \quad \lambda = 4 \cdot 3 \cdot l, \quad l \in Z, \\ \Omega(|q|_2), & |T|_2 \leq \frac{1}{2}, \quad \lambda = 4 \cdot 3 \cdot l, \quad l \in Z, \end{cases} \quad (69)$$

$$\Psi_p(q) = \begin{cases} \Omega(p^\nu |q|_p), & |T|_p \leq p^{-2\nu}, \quad |\lambda|_p \leq |3|_p^{1/2} p^{3\nu}, \quad \nu = 1, 2, 3, \dots, \\ \Omega(2^\nu |q|_2), & |T|_2 \leq 2^{-2\nu}, \quad |\lambda|_2 \leq 2^{3\nu-1}, \quad \nu = 1, 2, 3, \dots \end{cases} \quad (70)$$

Looking for the existence of the p -adic ground state in the form of the δ -function, we have to solve the integral equation

$$\begin{aligned} & \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p \left(-\frac{\lambda^2 T^3}{24} - \frac{T}{2} + \frac{\lambda q'' T}{4} + \frac{q''^2}{8T} \right) \\ & \times \int_{|q'|_p = p^\nu} \chi_p \left(\frac{q'^2}{8T} + \left(\frac{\lambda T}{4} - \frac{q''}{4T} \right) q' \right) dq' = \delta(p^\nu - |q''|_p). \end{aligned} \quad (71)$$

After the corresponding integration, for the left hand side of the previous equation, we obtain

$$\chi_p \left(-\frac{\lambda^2 T^3}{6} - \frac{T}{2} + \frac{\lambda q'' T}{2} \right).$$

By the very similar analysis for the parameter λ , we get $|\lambda|_p \leq p^{2-3\nu}$, and finally

$$\Psi_p(q) = \begin{cases} \delta(p^\nu - |q|_p), & |T|_p \leq p^{2\nu-2}, \quad |\lambda|_p \leq p^{2-3\nu}, \\ \delta(2^\nu - |q|_2), & |T|_2 \leq 2^{2\nu-1}, \quad |\lambda|_2 \leq 2^{-3\nu}, \end{cases} \quad (72)$$

where $\nu = 1, 0, -1, -2, \dots$ if $p \neq 2$, and $\nu = 0, -1, -2, \dots$ if $p = 2$.

This de Sitter model is adelic one. It allows eigenfunctions of the form (32), where for $p \in S$ states $\Psi_p(x_p)$ may be some solutions (70) and (72) with appropriately chosen $l \in Z$ in $\lambda = 3 \cdot 4 \cdot l$.

6.2. Model with a homogeneous scalar field

This is an adelic two-dimensional minisuperspace model with two decoupled degrees of freedom. On the basis of (57)-(60) and (61)-(62), the ground state with Ω -type functions is

$$\Psi_p(x, y) = \begin{cases} \Omega(|x|_p) \Omega(|y|_p), & |N|_p \leq 1, \quad \alpha = 4 \cdot 3 \cdot l_1, \quad \beta = 4 \cdot 3 \cdot l_2, \\ \Omega(|x|_2) \Omega(|y|_2), & |N|_2 \leq \frac{1}{2}, \quad \alpha = 4 \cdot 3 \cdot l_1, \quad \beta = 4 \cdot 3 \cdot l_2, \end{cases} \quad (73)$$

where $l_1, l_2 \in Z$, and also

$$\Psi_p(x, y) = \begin{cases} \Omega(p^\nu |x|_p) \Omega(p^\mu |y|_p), & |N|_p \leq p^{-2\nu}, \quad |N|_p \leq p^{-2\mu}, \\ \Omega(2^\nu |x|_2) \Omega(2^\mu |y|_2), & |N|_2 \leq 2^{-2\nu}, \quad |N|_2 \leq 2^{-2\mu}, \end{cases} \quad (74)$$

with $|\alpha|_p \leq |3|_p^{1/2} p^{3\nu}$, $|\beta|_p \leq |3|_p^{1/2} p^{3\mu}$ (if $p \neq 2$) and $|\alpha|_2 \leq 2^{3\nu-1}$, $|\beta|_2 \leq 2^{3\mu-1}$, where $\nu, \mu = 1, 2, 3, \dots$. As in the previous cases, we also investigate the existence of the vacuum state of the form $\delta(p^\nu - |x|_p) \delta(p^\mu - |y|_p)$. After the very similar calculations in Subsec. 6.1, we find p -adic wave function for the ground state

$$\Psi_p(x, y) = \begin{cases} \delta(p^\nu - |x|_p) \delta(p^\mu - |y|_p), & |N|_p \leq p^{2\nu-2}, \quad |N|_p \leq p^{2\mu-2}, \\ \delta(2^\nu - |x|_2) \delta(2^\mu - |y|_2), & |N|_2 \leq 2^{2\nu-1}, \quad |N|_2 \leq 2^{2\mu-1}, \end{cases} \quad (75)$$

with $|\alpha|_p \leq p^{2-3\nu}$, $|\beta|_p \leq p^{2-3\mu}$, and $|\alpha|_2 \leq 2^{-3\nu}$, $|\beta|_2 \leq 2^{-3\mu}$, where $\nu, \mu = 0, -1, -2, \dots$.

6.3. Anisotropic Bianchi Model with three scale factors

In this adelic case we start with metric²⁹

$$ds^2 = \sigma^2 \left[-\frac{N^2(t)}{a^2(t)} dt^2 + a^2(t) dx^2 + b^2(t) dy^2 + c^2(t) dz^2 \right]. \quad (76)$$

It leads to the action

$$S_v[a, b, c] = \frac{1}{2} \int_0^1 dt \left[-\frac{a}{N} (\dot{a}\dot{b}\dot{c} + a\dot{b}\dot{c} + \dot{a}b\dot{c}) - Nbc\lambda \right]. \quad (77)$$

By means of the substitution

$$x = \frac{bc + a^2}{2}, \quad y = \frac{bc - a^2}{2}, \quad \dot{z}^2 = a^2 \dot{b}\dot{c} \quad (78)$$

we obtain the quadratic classical action and propagator in the form

$$\bar{S}_v(x'', y'', z'', N; x', y', z', 0) =$$

$$- \frac{1}{4N} [(x'' - x')^2 - (y'' - y')^2 + 2(z'' - z')^2] - \frac{\lambda N}{4} [(x' + x'') + (y' + y'')], \quad (79)$$

$$\mathcal{K}_v(x'', y'', z'', N; x', y', z', 0) = \frac{\lambda_v(-2N)}{|4^{1/3}N|_v^{3/2}} \chi_v(-\bar{S}_v(x'', y'', z'', N; x', y', z', 0)). \quad (80)$$

By the above way, one gets the p -adic eigenstates

$$\Psi_p(x, y, z) = \begin{cases} \Omega(|x|_p) \Omega(|y|_p) \Omega(|z|_p), & |N|_p \leq 1, \quad |\lambda|_p \leq 1, \\ \Omega(|x|_2) \Omega(|y|_2) \Omega(|z|_2), & |N|_2 \leq \frac{1}{4}, \quad |\lambda|_2 \leq 2, \end{cases} \quad (81)$$

and

$$\Psi_p(x, y, z) = \begin{cases} \Omega(p^{\nu_1}|x|_p) \Omega(p^{\nu_2}|y|_p) \Omega(p^{\nu_3}|z|_p), \\ \Omega(2^{\nu_1}|x|_2) \Omega(2^{\nu_2}|y|_2) \Omega(2^{\nu_3}|z|_2), \end{cases} \quad (82)$$

with conditions: $|N|_p \leq p^{-2\nu_1}$, $|N|_p \leq p^{-2\nu_2}$, $|N|_p \leq p^{-2\nu_3}$, $|\lambda|_p \leq p^{3\nu_1}$, $|\lambda|_p \leq p^{3\nu_2}$, (if $p \neq 2$) and $|N|_2 \leq 2^{-2\nu_1-1}$, $|N|_2 \leq 2^{-2\nu_2-1}$, $|N|_2 \leq 2^{-2\nu_3-2}$, $|\lambda|_2 \leq 2^{3\nu_1+1}$, $|\lambda|_2 \leq 2^{3\nu_2+1}$, where $\nu_i = 1, 2, 3, \dots$.

For this model there also exist ground states

$$\Psi_p(x, y, z) = \begin{cases} \delta(p^{\nu_1} - |x|_p) \delta(p^{\nu_2} - |y|_p) \delta(p^{\nu_3} - |z|_p), \\ \delta(2^{\nu_1} - |x|_2) \delta(2^{\nu_2} - |y|_2) \delta(2^{\nu_3} - |z|_2), \end{cases} \quad (83)$$

with conditions: $|N|_p \leq p^{2\nu_1-2}$, $|N|_p \leq p^{2\nu_2-2}$, $|N|_p \leq p^{2\nu_3-2}$, $|\lambda|_p \leq p^{2-3\nu_1}$, $|\lambda|_p \leq p^{2-3\nu_2}$, and $|N|_2 \leq 2^{2\nu_1-3}$, $|N|_2 \leq 2^{2\nu_2-3}$, $|N|_2 \leq 2^{2\nu_3-3}$, $|\lambda|_2 \leq 2^{2-3\nu_1}$, $|\lambda|_2 \leq 2^{2-3\nu_2}$, where $\nu_1, \nu_2 = 1, 0, -1, -2, \dots$, $\nu_3 \in \mathbb{Z}$.

6.4. Some two dimensional models

There is a class of two-dimensional minisuperspace models which, after some transformations, obtain the form of two oscillators.^{30,31} These models are: the isotropic Friedmann model with conformally and minimally coupled scalar field, and the anisotropic vacuum Kantowski-Sachs model. For all these three models the corresponding action may be written as

$$S = \frac{1}{2} \int_0^1 dt N \left[-\frac{\dot{x}^2}{N^2} + \frac{\dot{y}^2}{N^2} + x^2 - y^2 \right], \quad (84)$$

i.e. this is the action for two linear oscillators, but one of them has negative energy. This classical action leads to the propagator

$$\mathcal{K}_p(y'', x'', N; y', x', 0) = \frac{1}{|N|_p} \chi_p \left(\frac{x''^2 + x'^2 - y'^2 - y''^2}{2 \tan N} + \frac{y' y'' - x' x''}{\sin N} \right). \quad (85)$$

The linear harmonic oscillator was analyzed from p -adic, as well as from the adelic point of view.^{2,5,12} One can show that in the p -adic region of convergence of analytic functions $\sin N$ and $\tan N$, which is $G_p = \{N \in Q_p : |N|_p \leq |2p|_p\}$, exist vacuum states $\Omega(|x|_p) \Omega(|y|_p)$, $\Omega(p^\nu |x|_p) \Omega(p^\mu |y|_p)$, $\nu, \mu = 1, 2, 3, \dots$, and also

$$\Psi_p(x, y) = \begin{cases} \delta(p^\nu - |x|_p) \delta(p^\mu - |y|_p), & |N|_p \leq p^{2\nu-2}, \quad |N|_p \leq p^{2\mu-2}, \\ \delta(2^\nu - |x|_2) \delta(2^\mu - |y|_2), & |N|_2 \leq 2^{2\nu-3}, \quad |N|_2 \leq 2^{2\mu-3}, \end{cases} \quad (86)$$

where $\nu, \mu = 0, -1, -2, \dots$. This is another example of p -adic and adelic minisuperspace cosmological model.

7. Concluding Remarks

In this paper, we find application of p -adic numbers in quantum cosmology very promising. It gives new and more complete insights into the space-time structure at the Planck scale.

In the Hartle-Hawking approach the wave function of a spatially closed universe is defined by Feynman's path integral method. The action is a functional of the gravitational and matter fields, and path integration is performed over all compact real four-metrics connecting two three-space states. Accordingly, the present adelic Hartle-Hawking proposal extends the ordinary one to the all corresponding compact p -adic metrics. Unfortunately, it does not lead to the adequate adelic generalization for a wide class of the minisuperspace models.

However, the consideration of minisuperspace models in the framework of adelic quantum mechanics gives the appropriate adelic generalization. Moreover, we can conclude that all the above adelic models lead to the new space-time picture in the vicinity of the Planck length.

Namely, for all the above adelic models there exist adelic ground states of the form

$$\Psi_S(q^1, \dots, q^n) = \prod_{\alpha=1}^n \Psi_{\infty}(q_{\infty}^{\alpha}) \prod_{p \in S} \Psi_p(q_p^{\alpha}) \prod_{p \notin S} \Omega(|q_p^{\alpha}|_p), \quad (87)$$

where $\Psi_{\infty}(q_{\infty}^{\alpha})$ are the corresponding real counterparts of the wave functions of the universe, and $\Psi_p(q_p^{\alpha})$ are proportional to $\Omega(p^{\nu}|q_p^{\alpha}|_p)$ or $\delta(p^{\nu} - |q_p^{\alpha}|_p)$ eigenfunctions with the corresponding normalization factors. Adopting the usual probability interpretation of the wave function (87), we have

$$|\Psi_S(q^1, \dots, q^n)|_{\infty}^2 = \prod_{\alpha=1}^n |\Psi_{\infty}(q_{\infty}^{\alpha})|_{\infty}^2 \prod_{p \in S} |\Psi_p(q_p^{\alpha})|_{\infty}^2 \prod_{p \notin S} \Omega(|q_p^{\alpha}|_p), \quad (88)$$

because $(\Omega(|q^{\alpha}|_p))^2 = \Omega(|q^{\alpha}|_p)$.

As a consequence of Ω -function properties, at the rational points q^1, \dots, q^n and $S = \emptyset$, we have

$$|\Psi(q^1, \dots, q^n)|_{\infty}^2 = \begin{cases} \prod_{\alpha=1}^n |\Psi_{\infty}(q^{\alpha})|_{\infty}^2, & q^{\alpha} \in Z, \\ 0, & q^{\alpha} \in Q \setminus Z. \end{cases} \quad (89)$$

This result leads to some discretization of minisuperspace coordinates q^{α} , because for all rational points density probability is nonzero only in the integer points of q^{α} . Keeping in mind that Ω -function is invariant with respect to the Fourier transform, this conclusion is also valid for the momentum space. Note that this kind of discreteness depends on adelic quantum state of the universe. When some p -adic states (for $p \in S$) are different from $\Omega(|q^{\alpha}|_p)$, then the above adelic discreteness becomes less transparent.

There is also some discreteness of the cosmological constant. Namely, the parameter λ is proportional to Λ , and consequently to the vacuum energy density. Since λ may have only integer values (see e.g. (69) for the de Sitter model), it follows that vacuum energy of the universe belongs to a discrete spectrum, which depends on its adelic quantum state.

It is worth noting that investigation of quantum properties of the de Sitter space is an actual subject. In particular, it has been argued³² and discussed³³ that it is possible that the Hilbert space of a quantum de Sitter space has a finite dimension. According to our results, the corresponding adelic Hilbert space is of the infinite dimension. Discreteness³⁴ of the Planck constant in the de Sitter space and the dS/CFT correspondence³⁵ have been also investigated.

Performing the integration in (88) over all the p -adic spaces, and having in mind that eigenfunctions should be normed to unity, one recovers the standard effective model over real space. However, if the region of integration is over only some parts of p -adic spaces then the adelic approach manifestly exhibits p -adic quantum effects. Since the Planck length is here the natural one, the adelic minisuperspace models refer to the Planck scale.

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